Approximation Algorithms for Steiner Tree Problems Based on Universal Solution Frameworks*

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Abstract. This paper summarizes the work on implementing few solutions for the Steiner Tree problem which we undertook in the PAAL project. The main focus of the project is the development of generic implementations of approximation algorithms together with universal solution frameworks. In particular, we have implemented Zelikovsky 11/6-approximation using local search framework, and 1.39-approximation by Byrka *et al.* using iterative rounding framework. These two algorithms are experimentally compared with greedy 2-approximation, with exact but exponential time Dreyfus-Wagner algorithm, as well as with results given by a state-of-the-art local search techniques by Uchoa and Werneck. The results of this paper are twofold. On one hand, we demonstrate that high level algorithmic concepts can be designed and efficiently used in C++. On the other hand, we show that the above algorithms with good theoretical guarantees, give decent results in practice, but are inferior to state-of-the-art heuristical approaches.

Keywords: Steiner tree,approximation algorithms, exact algorithms, iterative rounding, local search, greedy

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1 Introduction

Nowadays, working on state-of-the-art approximation algorithms requires the knowledge of many high-level tools and concepts. For example, a very successful line of research in approximation algorithms was based on iterative rounding idea [20], which lead to an approximation algorithm for the Steiner tree problem with the best theoretical approximation guarantee [4]. This means that implementing and testing such algorithms in practice is even harder, because one not only needs to understand these high level concepts but also is required to implement them. In our PAAL project[1] we have undertook a task to provide C++11 implementations of such high level tools including the iterative rounding, and local search frameworks. These frameworks have been implemented having the following design considerations in mind:

Easiness to use It should be always possible to build the approximation algorithm by implementing only functions, i.e., no definition of classes are needed.

Minimalism Our design minimizes number of functions one needs to write in order to achieve required results. Depending on selected optimization method programmer needs to provide only functions required for this method.

Speed Our library exploits benefits of static polymorphism supporting programs in both object-oriented and functional style. It enables compiler to use more sophisticated code optimization methods, function inlining, loop unrolling, branch prediction, etc. [10].

Loose coupling Library elements, as much as possible, do not depend on each other. It is possible to change behaviour of the solver by changing only one of its elements.

Extensibility One can add new solution strategies, e.g., search strategies, without modifying other elements of the framework.

In the case of local search these design assumption led to a framework, that on one hand, requires no class and much less function definitions when comparing it to other existing libraries like: ParadisEO[5], Metslib[21], or Easylocal [7]. On the other hand, the running time of our implementations, when counting only the time spend in the library functions, is at least 3 time smaller [7].

The aim of this paper is to report on the PAAL implementations of solutions for the Steiner tree problem. In this problem we are given an undirected graph $G = (V, E)$, with edge costs (weights) $c : E \to \mathbb{Q}_+$, and a subset of nodes $T \subseteq V$ (terminals), the *Steiner Tree* problem asks for a tree S spanning the terminals, of minimum cost $c(S) = \sum_{e \in S} c(e)$. Note that S might contain some other nodes, besides the terminals (Steiner nodes).

Steiner Tree problem is NP-hard and even APX-hard to approximate [6], i.e., approximating it better then $96/95 = 1.0105\ldots$ is NP-hard. During recent years

[1] The Practical Approximation Algorithms Library (PAAL) is a header-only, generic library consisting of approximation algorithms, data structures and several complete solutions for various optimization problems, implemented in C++11 available at `http://paal.mimuw.edu.pl`.

it has become a benchmark problem in approximation algorithms study. Three main techniques have been used to design approximation algorithms with good theoretical guarantees for this problem:

- greedy approach gives a 2-approximation [13],
- Zelikovsky introduced local search to obtain an 11/6-approximation [29],
- the best 1.39-approximation was proposed by Byrka *et al.* [4] and uses iterative rounding.[2]

We have implement the above 3 algorithms together with an exponential time exact algorithm given in [11]. The algorithm of Zelikovsky was implemented using our local search framework, whereas the algorithm by Byrka *et al.* was implemented using our iterative rounding framework. More details on the implementations and on how these frameworks aided us are given in the following sections of this paper. We have decided to compare our algorithms with results given by the state-of-the-art local search solution by Uchoa and Werneck [25].[3] For completeness of this paper we give some details of this solution in Section 6. The final section of this paper gives the result of our experimental study. A priori we were suspecting that the iterative rounding algorithm could deliver comparable results to the state-of-the-art heuristic solutions, as it was the case for the Minimum Bounded-Degree Spanning Tree (MBDST) problem [3]. However, the experiments show that this is not the case. On SteinLib [18] instances iterative rounding has an average approximation rato of 1.029, whereas the solution from [25] gives 1.01 on average. In other words, the additive error of our solution is on average 3 times higher. There are only few test cases, where iterative rounding found better answers. However, we note that the paper [25] considers 12 different local search algorithms, and only the best one of them visibly outperforms our iterative rounding implementation.

It appears that the main weakness of this iterative rounding solution is the need to generate all k-terminal components (i.e., k-terminal subtrees) from which an approximate Steiner tree can be build. On one hand, it seems that one really needs larger values of k to guarantee good approximation ratio. On the other hand, the generation of all such components is a bottleneck in the running time. When this procedure was implemented following exactly the description in [4] it took 98% of the running time needed by the algorithm. We have came up with several optimizations for this procedure, but even using them it still consumes 80% of the running time. We note that our implementation precedes the simplifications of Goemans *et al.* [15] to the algorithm of Byrka *et al.* [4]. However, these improvements are unlikely to have practical impact as the above bottleneck is still present there. Although, the implemented algorithms do not outperform state-of-the-art heuristics, our implementations have demonstrated that high level approximation algorithms can be implemented in an efficient and extendable way. As we have already mentioned our local search framework is

[2] For the full history of the theoretical studies of this problem please see [4].

[3] We would like to thank Renato Werneck for giving us these results, so we did not have to reimplement their solution.

easier and faster then alternative solutions, and PAAL library contains local search solutions for the following problems: traveling salesman problem, facility location, k-median, and capacitated facility location. On the other, our iterative rounding framework allowed us to easily implement solutions to the following additional problems: bounded-degree minimum spanning tree, generalised assignment, Steiner network, and tree augmentation.

The paper is composed as follows. The following four sections give the implementation details of greedy, Zelikovsky, Dreyfus-Wagner, and iterative rounding algorithms. Next, some details of the local search heuristics are given. Finally, Section 7 contains the description of our experiments.

2 Greedy 2-approximation

Let G^* be the metric closure of the graph G, and given a weighted graph H we denote by $\mathrm{MST}(H)$ the minimum spanning tree of H. It is well known that the minimum spanning tree of a metric closure of the graph restricted to terminals T (i.e., $\mathrm{MST}(G^*[T])$) is a 2-approximation of the Steiner Tree problem [26]. The time complexity of a naive implementation of the above algorithm equals $O(|T||E|\log(|V|))$, i.e., when one computes the distances from each terminal. PAAL implementation of this algorithm performs one run ofthe Dijkstra's algorithm starting from all terminals at once (see [22] for more details). This optimization helped us to reduce the time complexity of the algorithm to $O(|E|\log(|V|))$ (see [23] for the implementation).

3 11/6-approximation

The second algorithm implemented in PAAL is an 11/6-approximation by Zelikovsky [29]. We implemented the faster $O(|V| \cdot |E| + |T|^4)$ time complexity version of this algorithm. The algorithm is in a form of a local search, so we use PAAL's Local Search framework for the implementation. In particular we use the Hill Climbing primitive, i.e., we start with some solution and improve it as long as it is possible. PAAL provides framework for Hill Climbing consisting of three primitives (components):

- *State* – current solution,
- *Neighbourhood* – list of moves that can be applied to a state,
- *Gain* – difference between the value of a state after and before applying a move.

We present the outline of our implementation of Zelikovsky's algorithm. First the algorithm builds some initial data structures:

- Minimum Spanning Tree on the set of terminals,
- Voronoi regions of terminals: sets of Steiner vertices which are closer to a given terminal than to any other terminal,

– centers of terminal triples: for each triple of terminals we find its center, that is a Steiner vertex which minimizes the sum of distances to the terminals in the triple.

At each iteration the algorithm builds recursively a save matrix M (as defined in [29]) with rows and columns labeled by current terminals (some terminals are contracted during the algorithm). For any given pair of terminals T_1, T_2, the element of the save matrix $M[T_1, T_2]$, contains the cost of the most expensive edge on the cheapest path from T_1 to T_2. Next, by using Hill Climbing method, the algorithm iteratively improves the tree by adding new Steiner points to it. Each added point is a center of terminal triple and after adding it algortithm contracts the triple by setting costs of edges between tripple's vertices to 0. The pseudocode of the algorithm is given as Algorithm 1.

Algorithm 1 Pseudocode of Zelikovsky algorithm

$tree \leftarrow \text{MST}(G^*[T])$
$selected\_nonterminals \leftarrow \emptyset$
find Voronoi regions of all *terminal*s
for each $triple \in$ triples of terminals **do**
  $center(triple) \leftarrow$ center of $triple$ (as defined before)
  $cost(triple) \leftarrow$ sum of distances from the center to triple vertices
end for
loop
  $save \leftarrow$ save matrix of the $tree$
  $move \leftarrow$ triple which maximizes:

$$gain \leftarrow \max_{e \in triple} save(e) + \min_{e \in triple} save(e) - cost(triple)$$

  if $gain \leq 0$ **then** exit loop
  else
    contract $move$
    $selected\_nonterminals \leftarrow selected\_nonterminals + center(move)$
  end if
end loop
return $MST(G^*[T \cup selected\_nonterminals])$

Graph operations were implemented using Boost Graph Library [24]. Algorithm for computing Voronoi regions is implemented as a part of PAAL. Calculation of the *save* matrix is implemented recursively as it was presented in the original paper. Full C++ code can be found at [28].

4 Dreyfus-Wagner Algorithm

The Dreyfus-Wagner algorithm [11] finds an optimum solution to the Steiner Tree problem in exponential time (with respect to the number of terminals):

$O(3^{|T|}|V| + 2^{|T|}|V|^2)$. It will be also used to solve subproblems in the Iterative Rounding algorithm.

Our implementation is a straightforward recursive implementation of the Dreyfus and Wagner dynamic programming methods. For $X \subseteq T$ and $v \in V \setminus X$ we define $C(v, X)$ as the minimum cost of the Steiner tree spanning $X \cup \{v\}$ and $B(v, X)$ as the minimum cost of the Steiner tree spanning $X \cup \{v\}$, where v has degree at least two. The Dreyfus-Wagner algorithm is based on the following recursive formulas. The first formula comes from the fact, that given an optimal Steiner tree spanning $X \cup \{v\}$ in which the degree of v is at least two, we can split the tree at v into two subtrees: one spanning $Y \cup \{v\}$ and one spanning $(X \setminus Y) \cup \{v\}$, hence:

$$B(v, X) = \min_{\emptyset \subset Y \subset X} \{C(v, Y) + C(v, X \setminus Y)\}. \quad (1)$$

To get the second formula, let us consider the optimal Steiner tree spanning $X \cup \{v\}$, in which the degree of v is 1. In such case, the tree path form v leads either to a vertex $u \in X$ or a vertex $u \in V \setminus X$ of degree at least three, so:

$$C(v, X) = \min\{\min_{u \in X}\{C(u, X \setminus \{u\}) + d(u, v)\}, \min_{u \in V \setminus X}\{B(u, X) + d(u, v)\}\} \quad (2)$$

Where $d(u, v)$ is the shortest distance between u and v. In order to avoid calculating values $B(v, X)$ and $C(v, X)$ for the same states multiple times, we store the values for all previously processed states in a map. Full C++ implementation can be found at [1]

5 Iterative Rounding 1.39-approximation

Our last implementation is the LP-based randomized 1.39-approximation algorithm by Byrka *et al.* [4]. It is currently the best known approximation algorithm and is based on the Iterative Rounding technique introduced by Jain [17]. In the Iterative Rounding method we solve an LP-relaxation of the given problem, possibly obtaining a non-integer solution. We then iteratively round some LP variables according to problem-specific rules and resolve the modified LP, until we obtain an approximate solution to the original problem.

PAAL provides a generic framework for Iterative Rounding methods. Implementing an algorithm within this framework is based on providing the following primitives (components):

- *Init* – a functor responsible for initializing the LP for the given problem and initializing some additional data structures,
- *DependentRound* – a functor responsible for performing dependent LP rounding (rounding based on all of the LP variables values),
- *SetSolution* – a functor responsible for constructing the solution of the original problem.
- *SolveLP* – a functor responsible for solving the LP for the first time,

– *ResolveLP* – a functor responsible for resolving a previously solved and modified LP,
– *StopCondition* – a functor responsible for checking the stop condition for the Iterative Rounding main loop.

5.1 1.39-approximation Algorithm

The 1.39-approximation algorithm is based on an LP-relaxation known as the directed-component cut relaxation (DCR). First we need to give some necessary definitions afte [4].

Given a subset of terminals $T' \subseteq T$ and a terminal $t \in T'$ we define a *directed component* C on terminals T' with *sink* t as a minimum-cost Steiner tree on terminals T', with edges directed towards t. We call the terminals of a component C other then $\text{sink}(C)$ as $\text{sources}(C) = V(C) \cap T \setminus \{\text{sink}(C)\}$. We also denote the cost of a component as $c(C)$, the set of all components as C_n and we say that a component C *crosses* a set $U \subseteq T$ if C has at least one source in U and the sink outside U. By $\delta^+_{C_n}(U)$ we denote the set of directed components crossing U.

By selecting an arbitrary terminal r as a root, we can now formulate the DCR:

$$\begin{aligned} \text{minimize} \quad & \sum_{C \in C_n} c(C) x_C & & \\ \text{such that} \quad & \sum_{C \in \delta^+_{C_n}(U)} x_C \geq 1 & \forall U \subseteq T \setminus \{r\}, U \neq \emptyset & \qquad (3) \\ & x_C \geq 0 & \forall C \in C_n & \end{aligned}$$

As the size of the set C_n is exponential, we restrict it to a set C_k of directed components that contain at most k terminals (where k is a constant number). By replacing C_n with C_k in the DCR formulation, we obtain a k-DCR with polynomially many variables and exponentially many constraints. Despite the exponential number of constraints, the k-DCR can be solved in polynomial time using the so-called separation oracle (more details are given in the following sections).

Using the k-DCR formulation, we can give the pseudocode of the randomized 1.39-approximation algorithm:

5.2 Algorithm Implementation

To simplify the implementation we convert, without loss of generality, the input graph into its metric closure (complete weighted graph on the same nodes, with weights given by the shortest paths in the original graph).

The algorithm was implemented using the previously described PAAL Iterative Rounding framework. The main part of the implementation are the necessary framework primitives:

Algorithm 2 Pseudocode of the randomized 1.39-approximation

for $i = 1, 2, \ldots$ **do**
  $C_k \leftarrow$ all components on at most k terminals, each generated using Dreyfus-Wagner algorithm.
  Solve the k-DCR.
  Select one component C_i, where $C_i = C$ with probability $x_C / \sum_{C' \in C_k} x_{C'}$.
  Contract terminals of C_i into its sink.
  if Only one terminal remains **then**
    $i_{max} \leftarrow i$
    exit loop
  end if
end for
return $\bigcup_{i=1}^{i_{max}} C_i$

- *steiner_tree_init* – generates the C_k set and initializes the LP. To generate C_k we iterate over all subsets of T of size at most k and use the Dreyfus-Wagner algorithm to find the optimal Steiner tree on each subset.
- *steiner_tree_round_condition* – selects one random component C with probability $x_C / \sum_{C' \in C_k} x_{C'}$. Contracts terminals of C into its sink and updates the metric distances from the contracted node and reinitializes C_k and the LP (using *steiner_tree_init*).
- *steiner_tree_stop_condition* – checks if the number of remaining terminals is equal to 1.
- *steiner_tree_set_solution* – joins the sets of Steiner vertices from components selected in each phase.

The remaining primitives are the ones responsible for solving the LP. We detail their implementation in the following section.

Solving the LP As mentioned previously, the k-DCR LP has polynomially many variables but exponentially many constraints. The authors of the original paper [4] show, that the k-DCR can be reformulated into an equivalent polynomial sized LP by considering an nonsimultaneous multicommodity flow problem in an auxiliary directed graph. Despite its polynomial size, the equivalent formulation still has a large number of constraints: $O(k|T|^{k+1})$. We modify the approach from [4] and provide a *separation oracle* for the k-DCR and use it together with the *row generation* technique.

A *separation oracle* for an LP is an algorithm, which given a solution of the LP decides whether the solution is feasible or if not, returns a constraint violated by the solution. We can use the separation oracle to implement the *row generation* technique. This technique uses the following approach: solve an LP that contains only a subset of the constraints (the subproblem), let a basic optimal solution be x_0. If the oracle shows that x_0 satisfies all the constraints, then x_0 is a basic optimal solution of the original problem (since it is optimal for the subproblem, which is a relaxation, and feasible for the original). If, on

the other hand, the oracle finds a violated constraint, then add this constraint to the subproblem, and iterate the process.

Let us now describe the separation oracle for the k-DCR (it is similar to the oracle for the directed cut formulation described in [19]). Consider an auxiliary directed graph $G' = (V', E')$, where $V' = T \cup \{v_C | C \in C_k\}$. For every $C \in C_k$ we add an edge $e_C = (v_C, \text{sink}(C))$ and edges (u, v_C) for every $u \in \text{sources}(C)$. Let the edges e_C have weights x_C (the value of variable x_C in the solution being checked by the oracle), while the other edges have infinite weights.

Now consider a minimum directed cut in G' separating a vertex $v \in T \setminus \{r\}$ and r. Let the cut be $(T_1 \cup C_1, T_2 \cup C_2)$, where $T_1, T_2 \subset T$, $v \in T_1$, $r \in T_2$, $C_1, C_2 \subset \{v_C | C \in C_k\}$. If for some $u \in T_1$ there would exist a component C_u, such that $u \in \text{sources}(C_u)$ and $v_{C_u} \in C_2$, then the weight of the cut would be infinite and the cut would not be minimal. Thus for every $u \in T_1$ all components which contain u as a source must belong to C_1. The weight of the cut is equal to:

$$w(T_1 \cup C_1, T_2 \cup C_2) = \sum_{\{C \in C_1 | \text{sink}(C) \in T_2\}} x_C \tag{4}$$

It is easy to see, that for a given T_1 this weight is minimal when C_1 does not contain any components C', such that $\text{sources}(C') \cap T_1 = \emptyset$. In such case the weight of the cut is equal to the sum of x_C for components, which have at least one source in T_1 and the sink outside T_1 (components crossing T_1):

$$w(T_1 \cup C_1, T_2 \cup C_2) = \sum_{C \in \delta^+_{C_k}(T_1)} x_C \tag{5}$$

That way, we can describe the separation oracle for k-DCR: for all $v \in T \setminus \{r\}$ we check (using a polynomial minimum cut algorithm) if the weight of the minimum directed cut separating v from r is greater or equal to 1. If not, then the set C_1 defined by the minimum cut gives us a violated constraint.

We tried several heuristics to improve the running time of the row generation. First we tried to find the most violating constraint (that is, we iterate over all $v \in T \setminus \{r\}$ and select the smallest of all found cuts). We also tried to stop the search as soon as the first violated constraint was found (so we do not have to compute all $|T| - 1$ minimum cuts). The best running time was obtained by the following randomization: we choose a random permutation of $T \setminus \{r\}$ every time we use the oracle (as opposed to using the same permutation every time) and then we search until the first violated constraint is found.

We need to note that the row generation algorithm does not have a polynomial running time guarantee. An LP can be solved in polynomial time using a polynomial separation oracle [16] by the ellipsoid algorithm. However, because of the high complexity of the ellipsoid algorithm the row generation method works better in practice.

5.3 Components Generation

Experiments with the implementation have shown that generation of components is the bottleneck of the algorithm. To generate the set C_k, we iterate over all subsets of T with at most k elements and for each of them we run the Dreyfus-Wagner algorithm. Thus, the time complexity of this phase of the algorithm is $O(|T|^k \cdot (3^k|V| + 2^k|V|^2))$. Because of that, even for small values of k, for example $k = 4$, this phase is the bottleneck of the algorithm (for most instances component generation takes over 90% or even close to 100% of the total runtime).

To improve the algorithm running time we tested some optimizations for the component generation phase. First optimization comes from the fact, that we convert the graph into its metric closure. In such complete graph, for every subset T' of T we can find a tree with leafs form T' and not containing any other terminals (a component on T'). However, for some subsets T' there may not exist such a tree in the original graph. Because of that, we can ignore such subsets and decrease the number of calls to the Dreyfus-Wagner algorithm. To decide if a subset T' can produce a valid component, we need to check if there exists a path in the original graph between each two terminals form T' consisting only of non-terminals. After initial preprocessing in $O(|T| \cdot (|V| + |E|))$ time (running a BFS algorithm from each terminal), such check can be performed in $O(|T'|^2)$ time.

The speedup given by the above optimization depends heavily on the problem instance: it gives an improvement only in cases where many terminal pairs cannot be connected by a path consisting only of non-terminals. For such instances the optimization improved the algorithm running time by up to 5 times, however the component generation still remained the bottleneck of the implementation.

Components Generation Optimization The performance of the component generation is determined by the time spend in the Dreyfus-Wagner algorithm. In order to improve the component generation running times, we need to look into the details of that algorithm.

The Dreyfus-Wagner method is based on recursive caltulation of functions $B(v, X)$, $C(v, X)$ for certain nodes $v \in V$ and terminal subsets $X \subseteq T$ (of decreasing size). In the component generation phase of the IR algorithm, we repetitively run the Dreyfus-Wagner algorithm for different sets of terminals. However, it is easy to see, that functions B and C for many states (v, X) are calculated for more then one component. We can use this observation to implement the following optimization: we are going to store the values of B and C for all previously calculated states (from all previous components, not just the current one). Using the above optimization we were able to reduce the time complexity of component generation from $O(\sum_{i=2}^{k} \binom{|T|}{i}(3^i|V| + 2^i|V|^2))$ to $O(\sum_{i=2}^{k} \binom{|T|}{i}(2^i|V| + |V|^2))$.

This optimization gave a big improvement to the algorithm running time. For instances, which previously were solved in under 10 minutes, the optimization gave an average speedup of 6-10 times (depending on k) and up to 100 times speedup for some instances. It also increased the number of instances solved within the 10 minutes time limit by 10% for each tested parameter k. After the

optimization, the component generation phase took approximately 80% of the total running time as opposed to the previous 98-99%, however it still remained the bottleneck of the algorithm.

6 State-of-the-Art Multistart Local Search

We compare our approximation results with a results obtained using the best one out of the local search algorithms proposed by Uchoa and Werneck in [25], i.e., multistart heuristic (MS). For completeness of the paper we we will describe shortly their approach. We did not implement the heuristic by ourselves we just compare our results with results obtained from [25] authors.

The multistart heuristic: works in two phases. During the first phase it builds an initial MST using Shortest Path Heuristic (SPH). During the second phase it improves the MST using Hill Climbing for as long as improvement are possible using three types of moves. Authors restart the algorithm up to 100 times using different starting points for the SPH.

6.1 Shortest Path Heuristic

Shortest Path Heuristic greedily builds a initial Steiner tree in as shown in Algorithm 3. The tree built by the SPH depends on the choice of the starting

Algorithm 3 Shortest Path Heuristic

```
tree ← random node
while there are some terminals not in tree do
    t ← terminal not in tree that is closest to some node in tree
    tree ← tree+ shortest path from t to tree
end while
```

node, and on choices of nodes and paths made when there is a tie. Local Search algorithm builds only one tree for one starting node, but the whole procedure is used multiple times for randomly selected starting nodes.

6.2 Hill Climbing

After an initial tree is build using the SPH, it is improved using three types of moves via Hill Climbing method. The following moves are applied to the tree until no further improvement can be made.

Steiner Node Insertion. The first move type is the insertion of a Steiner node into the tree. We search for a vertex $v \notin tree$, such that the cost of MST($G[tree \cup \{v\}]$) is smaller than the cost of $MST(G[tree])$. If such node is found it means we have found a tree spanning all terminals which has a smaller cost than the previous one. We add this node to the set of selected Steiner points.

Key Path Exchange. A *key node* in a Steiner tree is a non-terminal node with a degree at least three. *Crucial nodes* of a Steiner tree are all terminals and key vertices. A *key path* is a path that connects two crucial nodes and has no internal crucial node.

The second kind of moves is a key path exchange. It removes a key path by splitting the tree into two connected components and then reconnects them using a new path. All improving key paths exchanges can be found by running Dijkstra's algorithm for each of the $O(|T|)$ key paths in the tree. The total running time of this algorithm when using Fibonacci heaps would be $O(|T|(|E| + |V| \log |V|))$ [27]. This time can be improved by using sequences of Voronoi diagrams as shown in [25], what gives an $O(|E| \log |V|)$ time algorithm for finding this improvement.

Key Vertex Elimination. Key nodes succinctly describe Steiner tree of a graph [12]. The third type of moves is the key vertex elimination. Let K be the set of key nodes in *tree*. We want to find a vertex v in K such that the cost of $\mathrm{MST}(G^*[K \cup T - v])$ is smaller then $\mathrm{MST}(G^*[K \cup T])$. Such vertex could be found by calculating MST for each of possible $O(|T|)$ vertices $v \in K$, however, it would give us an $O(|T|(|E| + |V| \log |V|))$ time algorithm [8]. In [25] authors improve running time of key vertex elimination to $O(|E| \log |V|)$.

7 Experiments

We have tested our implementations on data sets from SteinLib [18]. All tests were run with a timeout of 10 minutes. Most of the algorithms did not manage to solve all test cases in this time bound. The numbers of solved test cases are shown in Table 1. The table gives both the results for the IR implementation with and without the component generation optimization. In our further discussion we consider only the fastest (optimized) version of the algorithm.

All of our programs where compiled with `-O3` optimization option using gcc 4.8.1. We used a 24 core Intel Xeon CPU E5649@2.53GHz machine with Ubuntu 12.04 installed. The computer was equipped with 64GB of RAM. For solving the LP in the Iterative Rounding algorithm we used the GLPK LP solver [14]. We have not implemented Local Search on our own, since we got results from Uchoa and Werneck [25]. Their running times are always lower than 3 minutes and, unlike us, they implemented the algorithm using C#.

7.1 Comparison of the Main Algorithms

Figure 1 shows comparison of approximation ratios of four main algorithms: greedy 2-approximation, Zelikovsky algorithm, Local Search and IR for $k = 5$. We define approximation ratio as cost of algorithm solution divided by best known cost from SteinLib. The figure shows only those cases for which all algorithms were able to find solutions within the time limit, so there are 497 points.

Table 1. Numbers of solved SteinLib instances using different algorithms. IR optimized refers to the IR implementation with the component generation optimization.

Algorithm	Solved cases	Percent
Dreyfus Wagner	308	30%
Greedy 2 approx	1021	100%
Zelikovsky	964	94%
IR $k = 2$	770	75%
IR $k = 2$ optimized	858	84%
IR $k = 3$	695	68%
IR $k = 3$ optimized	762	75%
IR $k = 4$	550	54%
IR $k = 4$ optimized	617	60%
IR $k = 5$	389	38%
IR $k = 5$ optimized	514	50%
IR $k = 6$ optimized	452	44%

There are four histograms and we can easily see that for test cases from SteinLib Local Search achieves the best results, being slightly better than Iterative Rounding.

Every point on each of scatter plots represents one test case with its approximation ratios on x and y axis. We can see that there are cases for which Iterative Rounding performs better than Local Search.

Table 2 shows average approximation ratios and running times in seconds for each difficulty class, as defined in SteinLib [18].

Table 2. Average approximation ratios and running times in seconds of our algorithms for different SteinLib classes.

Class	Greedy		Zelikovsky		IR $k = 5$		Local Search
	ratio	time	ratio	time	ratio	time	ratio
euclidean	1.033	0.004	1.004	0.014	1.009	80.040	1.000
fst	1.037	0.002	1.017	0.125	1.030	73.642	1.001
hard	1.144	0.001	1.047	0.008	1.009	12.164	1.000
incidence	1.253	0.009	1.093	0.183	1.040	24.659	1.003
random	1.064	0.008	1.033	0.400	1.010	43.354	1.000
vlsi	1.052	0.007	1.012	0.146	1.011	70.459	1.000
average	1.140	0.006	1.052	0.176	1.028	46.983	1.001

7.2 Approximation Ratio of Iterative Rounding Depending on k

Figure 2 shows approximation ratios of IR depending on the value of k. On average, with increasing k approximation ratio gets closer to 1. There are some cases, where for bigger k IR gives worse results (we need to remember, however, that

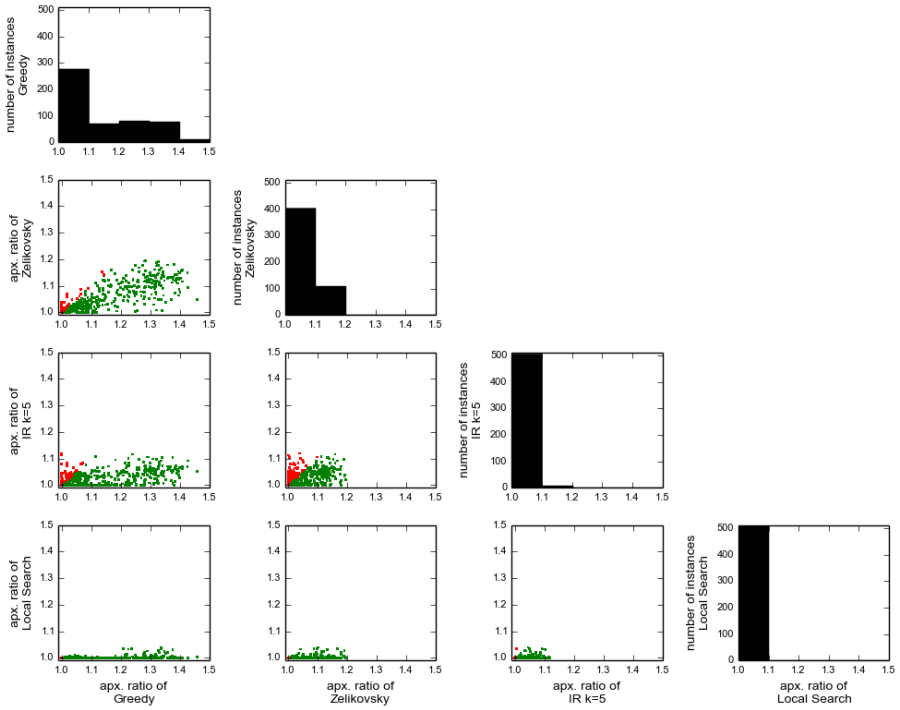

Fig. 1. Comparison of approximation ratios of following Steiner tree algorithms: Greedy, Zelikovsky, LS and IR ($k = 5$) on cases from SteinLib. There are four histograms on the plot, one for each algorithm. Histograms' bins show number of test cases (on y-axis) that fell into each interval of approximation ratios (on x-axis). Scatter plots compare performance of pairs of algorithms. Points on scatter plots represent test cases with approximation ratios achieved by compared algorithms on x and y-axis.

the IR is a randomized approximation algorithm). Table 3 shows average approximation ratios and average running times of Iterative Rounding for different values of k and different SteinLib dificulty classes.

7.3 Results

Comparing the 3 approximation algorithms implemented as a part of this work we see that for big enough parameters k ($k \geq 4$), the results returned by the IR algorithm are better then those returned by both the greedy and Zelikovsky algorithms.

On the other hand, the running times of the IR algorithm are much higher then for the other two algorithms. Also, while both the greedy and Zelikovsky algorithms were able to solve over 90% of SteinLib instances within our 10 minute time limit, the IR for $k = 4$ solved only 60% of the instances, and that number decreases for bigger values of k.

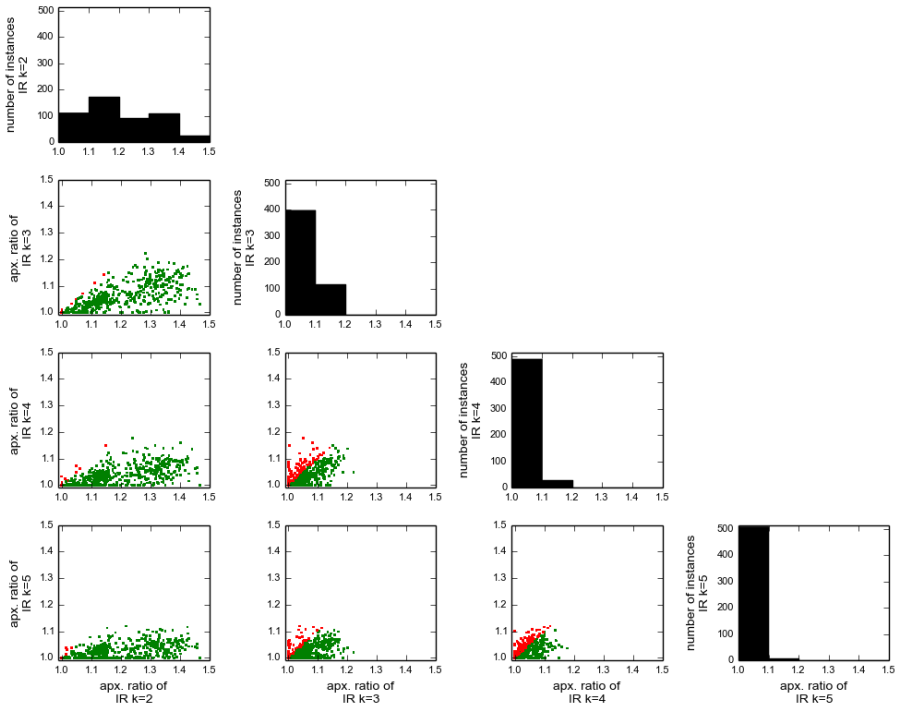

Fig. 2. Comparison of approximation ratios of Iterative Rounding depending on different values of k on cases from SteinLib. There are four histograms on the plot, one for each value of $k \in \{2, 3, 4, 5\}$. Histograms' bins show number of test cases (on y-axis) that fell into each interval of approximation ratios (on x-axis). Scatter plots compare performance of pairs of algorithms. Points on scatter plots represent test cases with approximation ratios achieved for compared values of k on x and y-axis.

The MS Local Search heuristics from [25] gives, on average, lower costs then all of the algorithms with theoretical guarantees we have implemented. It was also able to solve all SteinLib instances in under 3 minutes. Note, however, that the paper [25] gives 12 different versions of the local search and only the best one of them visibly outperforms our IR algorithm. The other ones give worse or comparable results.

Additionally, we have compared our IR results with few other papers. Our algorithm for $k > 4$ gives better results then the Tabu Search from [2], whereas it improves slightly over the results from [9] only on incidence class of SteinLib instances.

To summarize, the Iterative Rounding 1.39-approximation algorithm, which was the main interest of this paper, does seem to give approximation results that compare decently with other approaches. It is only outperformed by the best of Local Search implementations. However, despite good theoretical approximation

Table 3. Average approximation ratios and running times in seconds of the 1.39-approximation algorithm for different SteinLib classes and different values of k.

Class	IR $k=2$		IR $k=3$		IR $k=4$		IR $k=5$	
	ratio	time	ratio	time	ratio	time	ratio	time
euclidean	1.040	0.099	1.020	0.855	1.016	10.558	1.009	80.040
fst	1.126	7.537	1.044	10.317	1.033	28.813	1.030	75.684
hard	1.283	0.366	1.042	5.055	1.022	8.600	1.009	12.164
incidence	1.307	0.256	1.094	0.394	1.060	2.693	1.040	24.659
random	1.152	1.076	1.050	1.159	1.030	8.534	1.010	43.354
vlsi	1.083	1.027	1.024	1.104	1.014	17.915	1.011	70.459
average	1.202	2.211	1.063	3.111	1.041	12.307	1.028	47.527

ratio and decent experimental quality of solutions, it's running time is visibly higher then the one of the state-of-the-art heuristics like Local Search algorithms. Nevertheless, the most important goal of this study, i.e., to demonstrate that high-level approximation algorithm concepts can be implemented efficiently was accomplished successfully. Having implementations of these concepts available it is easier to continue the work on hybrid solutions that would combine the best aspects of different approaches and could potentially lead to better results.

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